Potential flow theory

Definitions

Streamlines	A line which is at all points tangential to the velocity vectors.
Streaklines	A line joining the istantaneous position of a succession of particles originating from one point
Pathlines	A line depicting the track of a single fluid particle.

For steady flow all three coincide, but not in unsteady flow eg smoke from a chimney issuing into unsteady air flow.

Rotational and irrotational flow

A particle is said to have zero rotation if the average of the angular velocity of two mutually perpendicular linear elements is zero. Examples of rotational flow is a forced vortex where velocity is proportional to the distance from the centre. A free vortex can be irrotational.

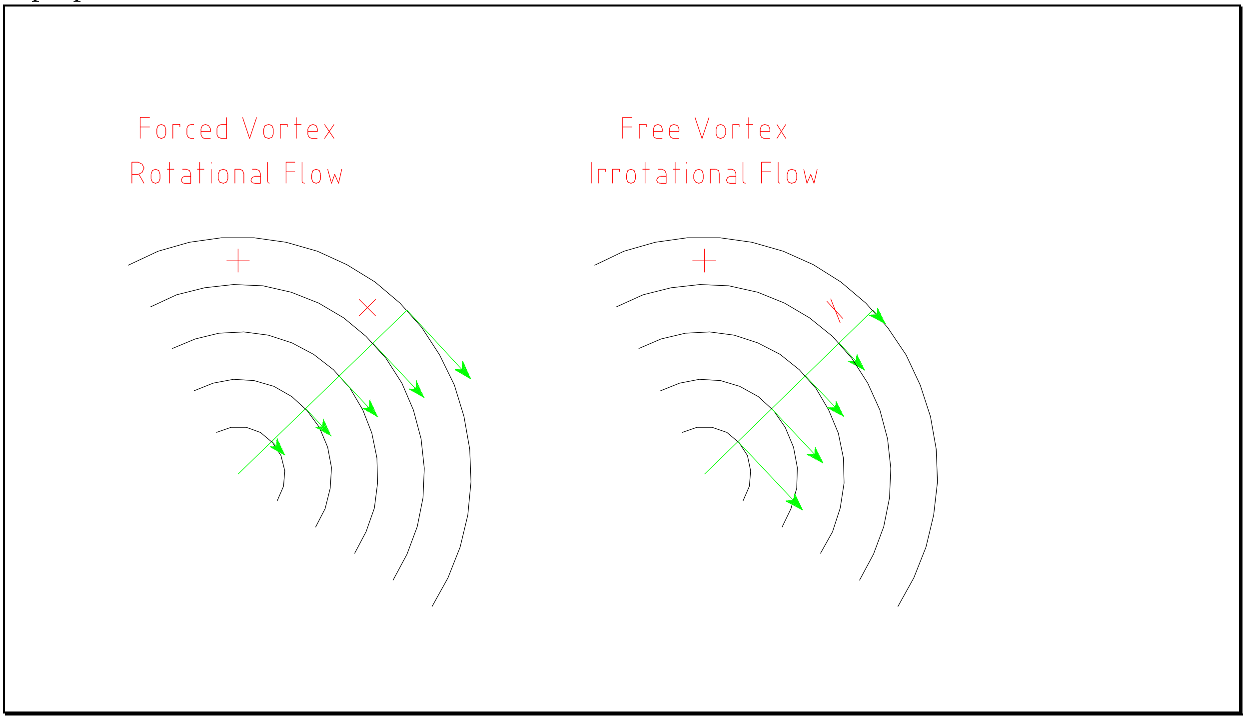


Figure 1 Example of Rotational and Irrotational Flow

A forced vortex occurrs when fluid is in a cylindrical container and the container is rotated about the longitudonal axis. The fluid at the centre is stationary and the fluid at the perimeter moves at the same speed as the container. For the free vortex, fluid at the perimeter is stationary and velocity elsewhere is **inversely** proportional to the radius. This causes a few problems at the centre ! The velocoity at the centre in fact, has become three dimensional so usually one must exclude that part from any calculation. Surprisingly the flow in a free vortex is mostly irrotational.

For the element in figure 2 the rotation about the z axis of AB is

$$\lim_{d\to 0}\frac{dq_1}{d} = \lim_{d\to 0}\frac{\frac{\partial v}{\partial x}\, dk\, d}{\partial x \partial t} = \frac{\partial v}{\partial x}$$

and of AD

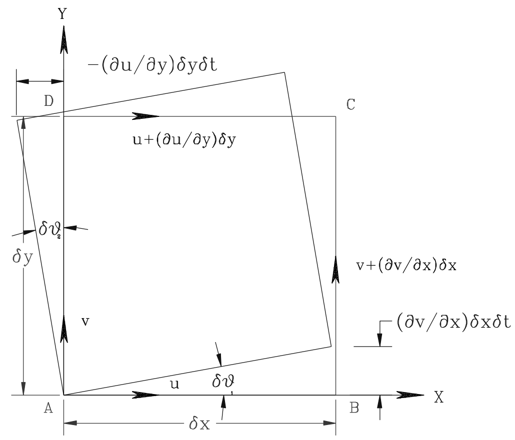


Figure 2 Definition of Rotation

$$\lim_{d\to 0}\frac{\partial q_2}{\partial t} = -\frac{\partial u}{\partial y}$$

The average of the two quantities is known as the **rotation**, ω

$$w = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

when ω is non zero the flow is said to be rotational and '**vorticity'** z (zeta) is said to exist

$$z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Thus the condition for two dimensional flow to be irrotational is

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \qquad \textbf{1}$$

In general for three dimensional irrotational flow

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \;;\; \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \;;\; \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \qquad \mathbf{2}$$

For a given point *P(x,y)* on a streamline, making an angle θ to the horizontal, the relationship

$$\frac{v}{u} = \tan q = \frac{dy}{dx}$$

and for three dimensional flow

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad \mathbf{3}$$

Potential function and Stream function

The kernel of potential flow theory is the definition of a velocity potential ϕ, such that

$$u = -\frac{\partial f}{\partial x}\left[\frac{\partial f}{\partial x}\right] \quad v = -\frac{\partial f}{\partial y}\left[\frac{\partial f}{\partial y}\right] \qquad \textbf{4}$$

where *u* is the velocity in the x direction and *v* is the velocity in the y direction.
As an alternative a stream function ψ, may be defined similarly as

$$u = -\frac{\partial y}{\partial y} \quad \left[\frac{\partial y}{\partial y}\right] \quad v = \frac{\partial y}{\partial x} \quad \left[-\frac{\partial y}{\partial x}\right] \qquad \textbf{5}$$

Other texts use a positive sign for the stream function and opposite signs for the stream function , terms in brackets).

For a streamline from equation (3)

$$vdx - udy = 0$$

$$\frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial y}dy = 0$$

$$\frac{\partial y}{\partial x}\,dx + \frac{\partial y}{\partial y}\,dy = 0 = dy$$

thus along any streamline ψ must be constant.

For a pair of streamlines the flow normal to a line, of length δl, joining both stream lines is

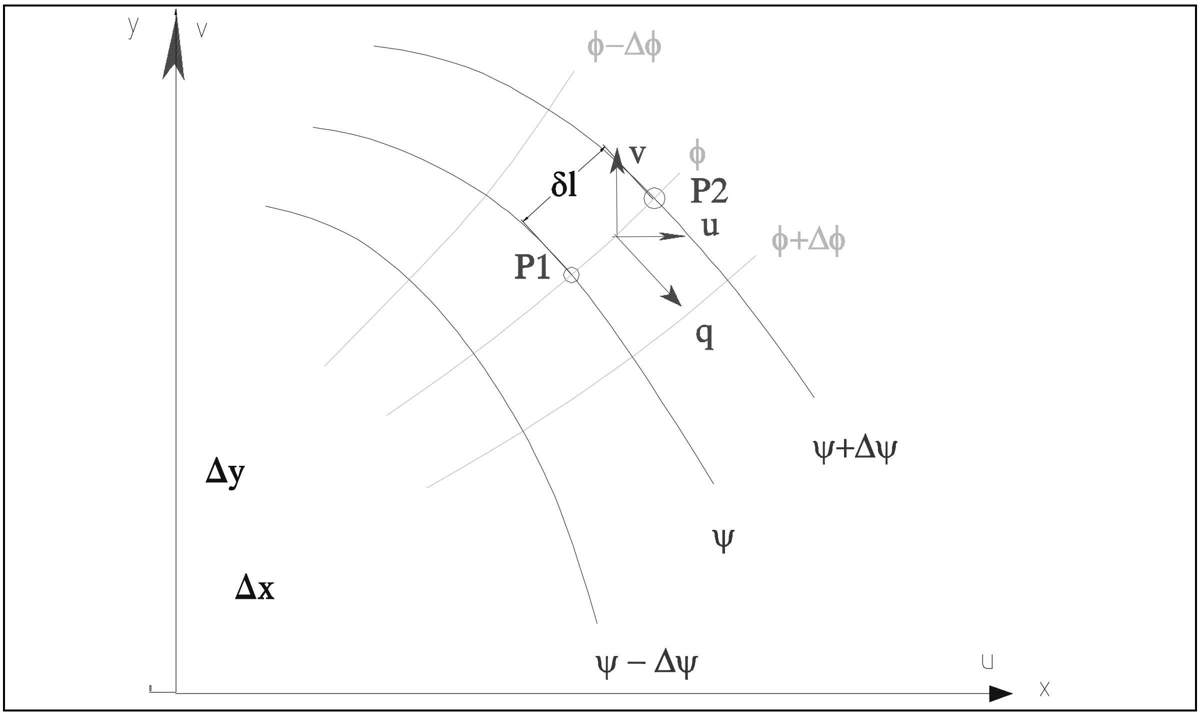
$$dQ = (u\sin q - v\cos q)\,d$$

$$= \left(u\frac{dy}{d} - v\frac{dx}{d}\right)d$$

$$= u\,dy - v\,dx$$

$$= -\frac{\partial y}{\partial y}\,dy - \frac{\partial y}{\partial x}\,dx = -dy$$

so that the flow between two streamlines is numerically equal to the difference in value of the stream function.



Similar relationships are obtained by considering the potential function. Equi-potential lines are a series of lines drawn at right angles to the streamlines. For steady flow along any potential line

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

$$-udx - vdy = 0$$

so that along a given equi-potential

$$\frac{dy}{dx} = -\frac{u}{v} \qquad \mathbf{6}$$

hence from equation (3) potential lines and stream lines are orthogonal.

If any of the expressions for *u* and *v* are substituted into equation (5) it is seen that

$$\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y} = 0$$

showing that ϕ satisfies the irrotational requirement. The existence of a potential function implies that flow is irrotational and the converse statement is also true that irrotationality implies the existence of a velocity potential. However steam functions are not restricted to irrotational flow.

Ideal flow assumption

The use of a velocity potential to model a flow implies certain assumptions must be made, these are -

1 The fluid behaves as an ideal fluid (inviscid and incompressible), and consequently the flow is irrotational since there can be no shear forces applied to an inviscid fluid.
2 There is no separation between fluid and solid boundaries.
3 The continuity equation must be satisfied. (see below)
4 Newtons second law of motion applies at all time and at every point. "*F=ma*"

Continuty Relationship

The general expression for three dimensional flow is

$$\frac{1}{r}\left(\frac{\partial r}{\partial t}+u\frac{\partial r}{\partial x}+v\frac{\partial r}{\partial y}+w\frac{\partial r}{\partial z}\right)+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \mathbf{8}$$

Thus for incompressible steady flow in two dimensions

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \mathbf{9}$$

Laplace Equation

By substitution of equation 4 into equation 9 yields

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0 \quad \text{or} \quad \nabla^2 f=0 \qquad \mathbf{10}$$

which is the well known Laplace equation.

Polar Coordinates

for a point *P(x,y)*, the following holds

$$x = r\cos\theta \; ; \; y = r\sin\theta \; ; \; r = \sqrt{x^2 + y^2} \; ; \; \theta = \tan^{-1}\frac{y}{x}$$

Two velocities may be defined, one radially outwards v_r and one tangential component, v_t. Recalling that the flowrate between two streamlines is numerically equal to the difference in the value of the streamfunction, then

$$-d\psi = v_r r d\theta - v_t dr$$

Also since φ is a function of *r* and θ then

$$d\psi = \frac{\partial \psi}{\partial \theta} d\theta + \frac{\partial \psi}{\partial r} dr$$

$$= \frac{1}{r}\frac{\partial \psi}{\partial \theta} r\, d\theta + \frac{\partial \psi}{\partial r} dr$$

thus it is evident that

$$v_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \; ; \; v_t = \frac{\partial \psi}{\partial r} \qquad \textbf{11}$$

In terms of the potential function

$$v_r = -\frac{\partial \phi}{\partial r} \; ; \; v_t = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} \qquad \textbf{12}$$

In terms of polar coordinates then the Laplace equation is

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} = 0 \qquad \textbf{13}$$

or

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} = 0 \qquad \textbf{14}$$

Equation of Motion

By considering the equilibrium of a particle of fluid, the application of Newton's second law of motion yields two dimensional equations of motion

$$
\begin{aligned}
-\frac{1}{\rho}\frac{\partial}{\partial x}(p+\rho g h) &= \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} \\
-\frac{1}{\rho}\frac{\partial}{\partial y}(p+\rho g h) &= \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}+u\frac{\partial v}{\partial x}
\end{aligned}
\qquad \mathbf{15}
$$

Now using the irrotationality

$$
\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \qquad \mathbf{16}
$$

yields a modification to equation 15

$$
\begin{aligned}
-\frac{1}{\rho}\frac{\partial}{\partial x}(p+\rho g h) &= \frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2) \\
-\frac{1}{\rho}\frac{\partial}{\partial y}(p+\rho g h) &= \frac{\partial v}{\partial t}+\frac{1}{2}\frac{\partial}{\partial y}(u^2+v^2)
\end{aligned}
\qquad \mathbf{17}
$$

and further rearrangement yields

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{p}{\rho}+gh+\frac{q^2}{2}-\frac{\partial \phi}{\partial t}\right) &= 0 \\
\frac{\partial}{\partial y}\left(\frac{p}{\rho}+gh+\frac{q^2}{2}-\frac{\partial \phi}{\partial t}\right) &= 0
\end{aligned}
\qquad \mathbf{18}
$$

where $q^2 = u^2 + v^2$. The terms in parenthesis are the same in both equations, and they are not functions of x or y since there derivatives are zero. Hence they define a function which depends only upon t, thus

$$
\frac{p}{\rho}+gh+\frac{q^2}{2}-\frac{\partial \phi}{\partial t}= f(t) \qquad \mathbf{19}
$$

For steady flow $\partial \phi/\partial t = 0$ and the function $f(t)$ becomes a constant, and equation 19 reverts to the Bernouilli equation. For wave theory this equation is the unsteady form of the Bernoulli equation, which is valid throughout the fluid. Dean and Dalrymple show that C(t) may be absorbed into the definition of the velocity potential and may be taken as zero so that

$$
\frac{p}{\rho}+gh+\frac{q^2}{2}-\frac{\partial \phi}{\partial t}=0 \qquad \mathbf{20}
$$

Potential flow examples

Uniform Flow

For a uniform flow field in the *x* direction the velocity potential is derived by setting - φ/∂x = *U* and integrating.

$$f = -Ux + C_1 \qquad \textbf{21}$$

or in polar coordinates *r*,θ where $x = r\cos\theta$ and $y = r\sin\theta$,

$$f = -Ur\ \cos\ q + C_2 \qquad \textbf{22}$$

In terms of the stream function the relationship is

$$y\ = -Uy + C_1 = -Ur\sin q + C_2 \qquad \textbf{23}$$

where the constants may readily be evaluated.

Point/Line Source/Sink

For a flowrate of *Q* we may choose a velocity potential such that the radial velocity, v_r and the tangential velocity, v_t are

$$v_r = -\frac{\partial f}{\partial r} = \frac{Q}{2pr} = \frac{m}{r} \qquad v_t = -\frac{1}{r}\frac{\partial f}{\partial q} = 0 \qquad \textbf{24}$$

Clearly the velocity potential is given as

$$f = -m\ln r + C_1\ \ =\ \ -m\ln(\sqrt{\left(x^2 + y^2\right)} + C_2 \qquad \textbf{25}$$

and satisfies the Laplace equation in polar coordinateseverywhere except at $r = 0$ which is clearly a singularity for this function. Note the stream function for the point source is $y\ = -mq + C_1$ and the strength of the source is $Q/2\pi$. *Q* is usually positive for flow **from** the source. If it is negative the singularity becomes a sink

Vortex (free)

If the streamlines and equipotentials in a source are reversed then the source becomes a vortex. The equations for a two dimensional vortex of strength Γ are

$$f = -\frac{\Gamma}{2p}\arctan\frac{y - y_0}{x - x_0} + \text{constant}$$

$$y\ = \frac{\Gamma}{2p}\ln R + \text{constant}$$

Note that in this case the angular velocity component is inversely proportional to the radius, in a forced vortex the angular velocity is proportional to the radius. The forced vortex is rotational!

Superposition

Source in uniform flow

Since the flow is irrotational the flows produced by two potential functions may be combined by simple addition of the potential functions and properties derived from the combined function. For example a source in a uniform x direction flow is

$$f = f_1 + f_2 = -m \ln r - Ur \cos \qquad \mathbf{26}$$

Source and Sink in uniform flow - Rankine Body

The source and sink are of equal strength and are sperated by a distance of 2a, with the origin midway between the two. The potential is

$$f = -Ux - \frac{Q}{4p}\left\{\ln\left[(x+a)^2 + y^2\right] - \ln\left[(x-a)^2 + y^2\right]\right\}$$

The length of the Rankine bode depends upon the relative magnitude of the source and sink and their spacing. If the two are moved closer together they become a circle when a = 0 but they than cancel each other out.

The Doublet

If the source and sink are at the same point, and to avoid mutual cancellation the strength is increased to infinity then the doublet results. Aquantity M is introduced such that

$$M = 2Qa = \text{constant}$$

The potential is thus

$$f = -\frac{Q}{4p}\ln\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}$$

As a becomes smaller we neglect the second order terms in a^2 so that

$$f = -\frac{Q}{4p}\ln\frac{1 + \dfrac{2ax}{x^2 + y^2}}{1 - \dfrac{2ax}{x^2 + y^2}}$$

The logarithm may be expanded in the form

$$\frac{1+e}{1-e} = 1 + 2e + \cdots$$

and

$$\ln(1+2e) = 2e - \frac{(2e)^2}{2} + \cdots$$

negelcting all higher order terms and for $e = 2ax^2/(x^2 + y^2)$

$$f = -\frac{Q}{4p}\left(\frac{4ax}{x^2 + y^2} + \cdots\right) \approx -\frac{M}{2p}\frac{x}{R^2}$$

which is the potential function for a doublet, where R is a polar coordinate. The stream function is given by

$$y = \frac{M}{2p}\frac{y}{R^2}$$

Adding a uniform flow to the doublet gives the flow around a cylinder of radius R_0
For the streamline with ψ=0, it is easy to arrange that

$$R^2 = \frac{M}{2pU} = R_0^2$$

Thus the potential and stream function for this flow are

$$f = -Ux\left(1+\frac{R_0^2}{R^2}\right) \qquad y = -Uy\left(1-\frac{R_0^2}{R^2}\right)$$

Potential Flow ~Problems

1 Show that the flowrate between two streamlines is numerically equal to the difference in value of the stream function on the respective streamlines.

2 A source of strength $\mu_1 = 2$ is located at (1,0) and another of strength $\mu_2 = 4$ is at (-1,0). Calculate the velocities at points (0,0),(0,1),(1,1). For y = 0 determine the value of x for which the u velocity is zero.[u=2,v=0;u=1,v=3;u=8/5,v=14/5;x=1/3]

3 A stream function is $\phi = -2xy+5$, show that the flow is irrotational and determine the potential function ($\psi = x^2 - y^2$)

4 Consider an incompressible flow with a stream function $\psi = x^2 + y^2$. Determine if the velocity potential for this flow exists. (no)

5 Plot the streamlines corresponding to the velocity potentials for a) $\phi = -xy$ and b) $\phi = -x^3 + 3xy^2$.

6 Consider a two-dimensional source of strength $q = 2\pi\mu$ per unit length and a sink of the same strength, both separated by a distance 2l. Prove that the streamlines are all circles passing through the source and sink

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